Int. J. Solids Structures Vol. 20, No. 2, pp. 129–139, 1984
Printed in Great Britain

0020-7683/84 $3.00 + .00

SOME ANNULAR DISC INCLUSION PROBLEMS IN ELASTICITY

[illegible]

Department of Civil Engineering, Carleton University, Ottawa, Ontario, Canada K1S 5B6

(Received 15 December 1982; in revised form 18 May 1983)

Abstract—This paper examines the problems related to the displacement and rotation of a rigid annular disc inclusion which is embedded in bonded contact with an isotropic elastic infinite space. The analysis of the [illegible]

1. INTRODUCTION

The class of problems related to the behaviour of flexible or rigid disc shaped inclusions [illegible] studies by Collins[1] and Keer[2] examine the problems of a rigid penny shaped inclusion embedded in bonded contact with an isotropic elastic solid. These studies were subsequently [illegible] inclusions and inhomogeneities embedded in elastic media are given by Mura[13], Willis[14] and Walpole[15].

This paper examines a series of axisymmetric and asymmetric problems related to an [illegible] translation of the inclusion in the x-direction. The rotationally symmetric deformations are induced by the torsion of the annular disc inclusion about the z-axis. By virtue of the [illegible]

Department of Civil Engineering, Carleton University, Ottawa, Ontario, Canada K1S 5B6

(Received 15 December 1982; in revised form 18 May 1983)

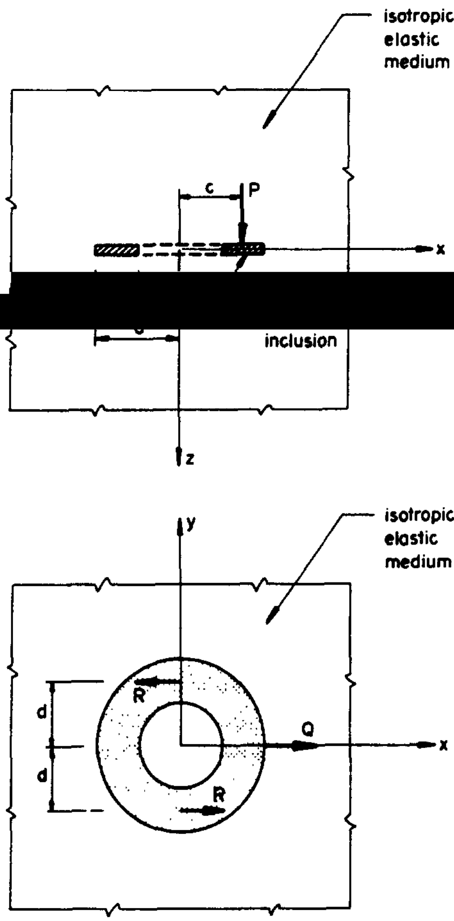

Fig. 1. Geometry of the annular disc inclusion and the resultant forces.

[redacted]

strengthen non-metallic or metallic matrices or increase the overall stiffness of a composite

[redacted]

In connection with the solution of the axisymmetric and asymmetric problems related to the embedded annular inclusion it is convenient to employ a formulation based on the strain potential approach of Love[21] and its extension to asymmetric problems proposed by [redacted]

Somigliana–Galerkin stress function[23, 24]. Proofs of the completeness of these represen- [redacted] function $\bar{\Psi}(r, \theta, z)$, i.e.:

$$\nabla^4\Phi(r, \theta, z) = 0; \nabla^2\Psi(r, \theta, z) = 0 \tag{1}$$

where

$$\nabla^4 = \nabla^2\nabla^2$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

[redacted]

We have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r \partial z} + \frac{2}{r}\frac{\partial\Psi}{\partial\theta} \tag{3a}$$

$$2Gu_\theta = -\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z} - 2\frac{\partial\Psi}{\partial r} \tag{3b}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{3c}$$

[redacted]
the components of the stress tensor σ are given by

[redacted]

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi - \frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\Psi \tag{4b}$$

[redacted]

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi - \frac{\partial^2\Psi}{\partial r\partial z} \tag{4d}$$

[redacted]

It may be noted that for axial symmetry $\Phi = \Phi(r, z)$ and $\Psi = 0$; thus the results (3) and (4) for

[redacted]

[redacted]

stresses, in the infinite space, about the plane $z = 0$. We may therefore restrict the analysis to a single halfspace region in which the plane $z = 0^+$ is subjected to appropriate mixed boundary [redacted]

(i) *For the rigid body translation in the z-direction*

[redacted]

$$u_r(r, 0^+) = 0; \; r \geq 0 \tag{5a}$$

[redacted]

$$\sigma_{zz}(r, 0^+) = 0; \; 0 < r < b. \tag{5d}$$

(ii) *For the rigid body rotation about the y-axis*

$$u_r(r, \theta, 0^+) = 0; \; r \geq 0 \tag{6a}$$

$$u_\theta(r, \theta, 0^+) = 0; \; r \geq 0 \tag{6b}$$

$$u_z(r, \theta, 0^+) = \Omega r \cos\theta; \; b \leq r \leq a \tag{6c}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; \; a < r < \infty \tag{6d}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; \; 0 < r < b. \tag{6e}$$

(iii) *For the rigid body rotation about the z-axis*

$$u_\theta(r, \theta, 0^+) = \omega r; \; b \leq r \leq a \tag{7a}$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; \; a < r < \infty \tag{7b}$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; \; 0 < r < b. \tag{7c}$$

(iv) *For the rigid body translation along the x-direction*

$$u_z(r, \theta, 0^+) = 0; \; r \geq 0 \tag{8a}$$

$$u_r(r, \theta, 0^+) = \delta \cos\theta; \; b \leq r \leq a \tag{8b}$$

$$u_\theta(r, \theta, 0^+) = -\delta \sin\theta; \; b \leq r \leq a \tag{8c}$$

$$\sigma_{rz} \sin\theta + \sigma_{\theta z} \cos\theta = 0; \; r \geq 0 \tag{8d}$$

$$\sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta = 0; \; a < r < \infty \tag{8e}$$

$$\sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta = 0; \; 0 < r < b. \tag{8f}$$

The boundary conditions (8d), (8e) and (8f) relate to the traction vectors which act on the plane $z = 0^+$ along the y and x directions, respectively.

[redacted]

solutions for Φ and Ψ take the following forms.

(i) *For the rigid body translation in the z-direction*

$$\Phi(r, z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{9a}$$

$$\Psi(r, z) = 0. \tag{9b}$$

(ii) *For the rigid body rotation about the y-axis*

$$\Phi(r, \theta, z) = \left\{\int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \cos\theta \tag{10a}$$

$$\Psi(r, \theta, z) = \left\{\int_0^\infty \xi C(\xi)\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \sin\theta. \tag{10b}$$

(iii) *For the rigid body rotation about the z-axis*

$$\Phi(r, \theta, z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_1(\xi r)\, d\xi \tag{11a}$$

$$\Psi(r, \theta, z) = 0. \tag{11b}$$

(iv) *For the rigid body translation along the x-direction*

$$\Phi(r, \theta, z) = \left\{\int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \cos\theta \tag{12a}$$

$$\Psi(r, \theta, z) = \left\{\int_0^\infty \xi C(\xi)\, e^{-\xi z} J_1(\xi r)\, d\xi\right\} \sin\theta \tag{12b}$$

For convenience, we define the nth order Hankel operator as follows:

$$H_n[f(\xi); r] = \int_0^\infty \xi f(\xi) J_n(\xi r)\, d\xi. \tag{13}$$

[illegible] the displacement and stress components given by (3a–c) and (4a–f) it can be shown that the mixed boundary conditions (5)–(8) reduce to sets of triple integral equations for an unknown function $R_n(\xi)(n = 1,2,3,4)$.

(i) For the rigid body displacement of the annular disc inclusion in the z-direction we have

$$H_0[R_1(\xi); r] = 0; 0 < r < b \tag{14a}$$

$$H_0[\xi^{-1} R_1(\xi); r] = -\frac{2\delta(1-\nu)}{(3-4\nu)}; b \le r \le a \tag{14b}$$

$$H_0[R_1(\xi); r] = 0; a < r < \infty. \tag{14c}$$

[redacted]

$$H_1[\xi^{-1}R_2(\xi); r] = 0; 0 < r < b \tag{15a}$$

$$H_1[R_2(\xi); r] = -\frac{2\Omega r(1-\nu)}{(3-4\nu)}; b \le r \le a \tag{15b}$$

$$H_1[\xi^{-1}R_2(\xi); r] = 0; a < r < \infty. \tag{15c}$$

[redacted]

$$H_1[R_3(\xi); r] = 0; 0 < r < b \tag{16a}$$

$$H_1[\xi^{-1}R_3(\xi); r] = \omega r; b \le r \le a \tag{16b}$$

[redacted]

(iv) For the lateral translation of the annular disc inclusion along the x-direction we have

$$H_1[R_4(\xi); r] = 0; 0 < r < b \tag{17a}$$

$$H_1[\xi^{-1}R_4(\xi); r] = -\frac{4\Delta(1-\nu)}{(7-8\nu)}; b \le r \le a \tag{17b}$$

$$H_1[R_4(\xi); r] = 0; a < r < \infty. \tag{17c}$$

The sets of triple integral equations defined by (14)–(17) can be solved by employing a variety [redacted] the method of solution proposed by Williams[20]. In its general form, the triple system can be written as

$$H_n[R(\xi); r] = 0; 0 < r < b \tag{18}$$

$$H_n[\xi^{-1}R(\xi); r] = f(r); b \le r \le a \tag{19}$$

$$H_n[R(\xi); r] = 0; a < r < \infty. \tag{20}$$

[redacted]

$$H_n[R(\xi); r] = g(r); b < r < a. \tag{21}$$

From the Hankel inversion theorem we have

$$R(\xi) = \int_a^b rg(r)J_n(\xi r)\,dr. \tag{22}$$

Using this result in (19) we obtain

$$\int_a^b ug(u)K_0(u, r)\,du = f(r); b \le r \le a \tag{23}$$

where

$$K_0(u, r) = u\int_0^\infty J_n(\xi r)J_n(\xi u)\,d\xi. \tag{24}$$

We define the functions $g_1(u)$ and $g_2(u)$ such that

$$g_1(u)+g_2(u)=\begin{cases}0 & ;0\le r<b\\ g(u) & ;b\le r\le a\\ 0 & ;a<r<\infty\end{cases} \tag{25}$$

and assume that $f(r)$ admits expansions of the form

[redacted]

$$f_2(r)=\sum_{n=-\infty} a_n r^n;\, b<r<\infty. \tag{27}$$

From the representations (24)–(27) it follows that the integral equation (23) reduces to two integral equations

$$\int_0^\infty uK_0(u,r)g_1(u)\,\mathrm{d}u=f_1(r);\,0<r<a \tag{28}$$

$$\int_0^\infty uK_0(u,r)g_2(u)\,\mathrm{d}u=f_2(r);\,b<r<\infty. \tag{29}$$

[redacted]

$$\int_t^\infty tK_0(r,t)g(t)\,\mathrm{d}t=4r^{-n}\int_0^r\frac{s^{2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_s^\infty\frac{t^{1-n}g(t)\,\mathrm{d}t}{(t^2-s^2)^{1/2}};\,0<r<\infty \tag{30}$$

$$\int_0^\infty tK_0(r,t)g(t)\,\mathrm{d}t=4r^{n}\int_r^\infty\frac{s^{-2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_0^s\frac{t^{1+n}g(t)\,\mathrm{d}t}{(s^2-t^2)^{1/2}};\,0<r<\infty. \tag{31}$$

Using these results, the integral equations (28) and (29) can be expressed in the form

$$4r^{-n}\int_s^r\frac{s^{2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_s^\infty\frac{t^{1-n}g_1(t)\,\mathrm{d}t}{(t^2-s^2)^{1/2}}=f_1(r);\,0<r<a \tag{32}$$

$$4r^{n}\int_r^\infty\frac{s^{-2n}\,\mathrm{d}s}{(s^2-r^2)^{1/2}}\int_0^s\frac{t^{1+n}g_2(t)\,\mathrm{d}t}{(s^2-t^2)^{1/2}}=f_2(r);\,b<r<\infty. \tag{33}$$

The next step is to define unknown functions $S_i(r)$, $T_i(r)$ and $C_i(r)$ $(i=1,2)$ such that

[redacted]

$$r^{-n}\int_0^r\frac{t^{1+n}g_2(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}}=\begin{cases}-T_2(r);\,0<r<b\\ S_2(r);\,b<r<\infty\end{cases} \tag{35}$$

$$4r^{-n}\int_0^r\frac{s^nC_1(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}}=f_1(r);\,0<r<a \tag{36}$$

$$4r^{n}\int_r^\infty\frac{s^{-n}C_2(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}}=f_2(r);\,b<r<\infty \tag{37}$$

$$S_i(r)=C_i(r). \tag{38}$$

The four integral equations (35)–(38) can be inverted to give

[redacted]

$$g_2(t) = \frac{2}{\pi} t^{-n-1} \frac{d}{dt}\left[-\int_0^b \frac{u^{n+1}T_2(u)\,du}{(t^2-u^2)^{1/2}} + \int_b^t \frac{u^{n+1}S_2(u)\,du}{(t^2-u^2)^{1/2}}\right] \tag{40}$$

$$C_1(r) = \frac{1}{2\pi r^n}\frac{d}{dr}\int_0^r \frac{u^{n+1}f_1(u)\,du}{(r^2-u^2)^{1/2}};\ 0<r<a \tag{41}$$

$$C_2(r) = -\frac{r^n}{2\pi}\frac{d}{dr}\int_r^\infty \frac{u^{1-n}f_2(u)\,du}{(u^2-r^2)^{1/2}};\ b<r<\infty. \tag{42}$$

[redacted]

$$T_1(r) = l_1(r) + \frac{n!}{r^n\sqrt{(\pi)}\Gamma(n+(3/2))}\int_0^b \frac{u^{n+1}T_2(u)\,{}_2F_1((1/2),n;n+(3/2);(u^2/r^2))\,du}{(r^2-u^2)}\quad ;a<r<\infty \tag{43}$$

[redacted]

where ${}_2F_1$ is a hypergeometric function and $l_1(r)$ and $l_2(r)$ are given by

$$l_1(r) = -\frac{2}{\pi r^n}\int_0^r \frac{t^{2n}\,dt}{(r^2-t^2)^{1/2}}\frac{d}{dt}\int_t^a \frac{u^{1-n}S_1(u)\,du}{(u^2-t^2)^{1/2}} \tag{45}$$

[redacted]

The integral equations (43) and (44) can be solved, by using iterative techniques, to yield expressions for $T_i(r)$; these in turn can be used in (39) and (40) to generate the expressions for $g_i(t)$. Specific results derived from the method are outlined below.

(i) For example, for the rigid body displacement of the disc inclusion in the axial direction we have

$$n=0;\ f(r)=A=\text{const.};\ f_1(r)=A;\ f_2(r)=0. \tag{47}$$

From (38), (41) and (42) we have

$$C_1 = S_1 = \frac{A}{2\pi};\ C_2 = S_2 = 0. \tag{48}$$

Making use of (43)–(46) we find that

$$l_1(br) = \frac{1}{\pi^2}\left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + 0(\lambda^9)\right]$$

$$l_2(ar) = 0 \tag{49}$$

$\lambda = b/a$ and $0(\lambda^n)$ is the Landau symbol.

[redacted]

$$T_1(ar) = \frac{2\lambda}{\pi^2}\left[\frac{1}{r^2}\left(\frac{\lambda}{3}+\frac{\lambda}{15}+\frac{4\lambda}{27\pi^2}+\frac{\lambda}{35}+\frac{92\lambda}{675\pi^2}\right)\right.$$

$$\left.+\frac{1}{r^4}\left(\frac{\lambda^3}{5}+\frac{\lambda^5}{21}+\frac{4\lambda^6}{45\pi^2}\right)+\frac{\lambda^5}{7\pi^6}+0(\lambda^7)\right];\ 1<r<\infty \tag{50}$$

$$T_2(br) = \frac{1}{\pi^2}\left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + \frac{4}{\pi^2}\left\{\left(\frac{\lambda^4}{9} + \frac{14\lambda^6}{225} + \frac{4\lambda^7}{81\pi^2} + \frac{29\lambda^8}{735}\right) r + \left(\frac{\lambda^6}{15} + \frac{22\lambda^8}{525}\right) r^3 + \frac{\lambda^2 r^5}{21}\right\} + O(\lambda^9)\right]; 0 < r < 1. \tag{51}$$

These results can be used to develop the relevant expression for $g(t)(= g_1(t) + g_2(t))$.

[redacted]

where B is a constant

$$C_1(r) = S_1(r) = 2Br;\ C_2(r) = S_2(r) = 0.$$

The corresponding expressions for $T_1(ar)$ and $T_2(br)$ take the forms:

$$T_1(ar) = \frac{32Ba\lambda^5}{45\pi^2}\left[\frac{1}{r^3}\left(1 + \frac{2\lambda^2}{7}\right) + \frac{6\lambda^2}{7r^5} + O(\lambda^4)\right]; 1 < r < \infty \tag{52}$$

[redacted]

Similar results can be derived for the problems which relate to rotation of the disc inclusion [redacted]

applications.

(i) *Rigid body translation in the z-direction*

[redacted]

y-axis. Considering the axisymmetric problem we have

$$P = 2\pi \int_b^a r[\sigma_{zz}(r, 0^-) - \sigma_{zz}(r, 0^+)]\, dr \tag{54}$$

where $\sigma_{zz}(r, 0^+)$ and $\sigma_{zz}(r, 0^-)$ refer to the normal interface stresses which act on the faces of the [redacted]

Considering (47) we can set $A = -2\delta(1-\nu)/(3-4\nu)$; consequently (55) yields

$$P = \frac{64(1-\nu)Ga\delta}{(3-4\nu)}\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + O(\lambda^9)\right]. \tag{56}$$

We note that as $\lambda \to 0$, (56) reduces to the classical result for the solid disc inclusion derived by Collins[1], Kanwal and Sharma[19] and Selvadurai[18].

(ii) *Rigid body rotation about the* y-*axis*

The resultant moment $M_0 = Pc$ is given by

$$M_0 = \pi \int_b^a r[\sigma_{zz}(r, \theta, 0^+) - \sigma_{zz}(r, \theta, 0^-)]\, dr. \quad (57)$$

Again $\sigma_{zz}(r, \theta, 0^+) = -\sigma_{zz}(r, \theta, 0^-)$ and

$$M_0 = -4\pi G \int_b^a r^2 g(r)\, dr. \quad (58)$$

[illegible]

$$M_0 = \frac{64(1-\nu)G\Omega a^3}{[illegible]}\left[1 - \frac{16\lambda^5}{[illegible]} - \frac{64\lambda^7}{[illegible]} + 0(\lambda^9)\right] \quad (59)$$

Again as $\lambda \to 0$, (59) reduces to the result for the solid inclusion given by Selvadurai[7].

[illegible]

(iii) *Rigid body rotation about the* z-*axis*

The forces R act in the plane of the disc inclusion. These forces are equivalent to a resultant torque $T(=2Rd)$ which acts about the z-axis. The magnitude of T is given by

$$T = 2\pi \int_b^{[illegible]} r^2[\sigma_{\theta z}(r, 0^-) + \sigma_{\theta z}(r, 0^+)]\, dr \quad (60)$$

Using the results derived in the previous sections it can be shown that

$$T = \frac{32Ga^3\omega}{3}\left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + 0(\lambda^9)\right]. \quad (61)$$

The result (61) is in agreement with analogous results derived by Collins[28] for the Reissner–Sagoci problem for an annular punch.

(iv) *Rigid body translation along the* x-*axis*

The application of the force Q causes a rigid body translation (Δ) of the annular disc inclusion along the x-direction.

$$Q = \int_b^a \int_0^{2\pi} [T_x(r, \theta, 0^+) + T_x(r, \theta, 0^-)] r\, dr\, d\theta. \quad (62)$$

The load–displacement relationship takes the form

$$Q = \frac{64(1-\nu)Ga\Delta}{(7-8\nu)}\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + 0(\lambda^9)\right]. \quad (63)$$

As $\lambda \to 0$, (63) reduces to the results given by Keer[2], Kassir and Sih[3] and Selvadurai[9] for the lateral translation of the embedded solid disc inclusion.

REFERENCES

1. W. D. Collins, Some axially symmetric stress distributions in elastic solids containing penny shaped cracks, I. Cracks [illegible]

[illegible]

3. M. K. Kassir and G. C. Sih, Some three-dimensional inclusion problems in elasticity theory. [illegible] 225–241 (1968).

[illegible]

[redacted]
medium. *Solid Mech. Arch.* 4, 163–172 (1979).
6. A. P. S. Selvadurai, An energy estimate of the flexural behaviour of a circular foundation embedded in an isotropic elastic medium. *Int. J. Num. Anal. Meth. Geomech.* 3, 285–292 (1979).
7. A. P. S. Selvadurai, The eccentric loading of a rigid circular foundation embedded in an isotropic elastic medium. *Int. J. Num. Anal. Meth. Geomech.* 4, 121–129 (1980).
8. A. P. S. Selvadurai, The displacements of a flexible inhomogeneity embedded in a transversely isotropic elastic medium. *Fibre Sci. Tech.* 14, 251–259 (1981).
9. A. P. S. Selvadurai, Asymmetric displacements of a rigid disc inclusion embedded in a transversely isotropic elastic medium of infinite extent. *Int. J. Engng Sci.* 18, 979–986 (1980).
10. A. P. S. Selvadurai, Betti's reciprocal relationships for the displacements of an elastic infinite space bounded internally by a rigid inclusion. *J. Structural Mech.* 9, 199–210 (1981).
11. A. P. S. Selvadurai, On the interaction between an elastically embedded rigid inhomogeneity and a laterally placed
[redacted]
12. A. P. S. Selvadurai, Axial displacement of a rigid elliptical disc inclusion embedded in a transversely isotropic elastic solid. *Mech. Res. Commun.* 9, 39–45 (1982).
[redacted]
16. R. H. Edwards, Stress concentration around spheroidal inclusions and cavities. *J. Appl. Mech.* 18, 19–30 (1951).
17. A. I. Luré, Elastostatic problem for a triaxial ellipsoid. *Mech. Tverdogo Tela.* 1, 80–83 (1967).
[redacted]
603–612 (1976).
19. R. P. Kanwal and D. L. Sharma, Singularity methods for elastostatics. *J. Elasticity* 6, 405–418 (1976).
20. W. E. Williams, Integral equation formulation of some three part boundary value problems. *Proc. Edin. Math. Soc.*
[redacted]
21. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th Edn. Dover Publications, New York (1944).
22. R. Muki, Asymmetric problems of the theory of elasticity for a semi-infinite solid and a thick plate, in *Progress in Solid*
[redacted]
23. M. E. Gurtin, The linear theory of elasticity. In *Mechanics of Solids II, Encyclopaedia of Physics* (Edited by S. Flugge).
[redacted]
25. C. Truesdell, Invariant and complete stress functions for general continua. *Arch. Rat. Mech. Anal.* 4, 1–29 (1960).
26. J. C. Cooke, Triple integral equations. *Quart. J. Mech. Appl. Math.* 16, 193–203 (1963).
[redacted]
28. W. D. Collins, On the solution of some axisymmetric boundary value problems by means of integral equations. IV. Potential problems for a circular annulus. *Proc. Edin. Math. Soc.* 13, 235–246 (1963).
29. D. L. Jain and R. P. Kanwal, Three part boundary value problems in potential and generalized axisymmetric potential theories. *J. d'Anal. Math.* 25, 107–158 (1972).
30. I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*. North Holland, Amsterdam (1966).
[redacted]

APPENDIX A

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{J_{n-1/2}(ps)s^{n+(1/2)}\,ds}{(r^2-s^2)^{1/2}} \tag{A1}$$

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} r^n \int_r^\infty \frac{J_{n+(1/2)}(ps)s^{-n+(1/2)}\,ds}{(s^2-r^2)^{1/2}} \tag{A2}$$

$$\int_0^\infty pJ_{n+(1/2)}(ps)J_{n+(1/2)}(pt)\,dp = \frac{\delta^*(s-t)}{(st)^{1/2}} \tag{A3}$$

where δ^* is the Dirac delta function.